Mutual Impedance Between Probes in a Circular Waveguide

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Abstract—**The general formulas of mutual impedance between two probes arbitrarily located in a circular waveguide are given by means of a dyadic Green's function (DGF) and reaction concept. The waveguide is semi-infinite. The reflection coefficient at the terminal plane is Γ. The lengths, feeding points, and orientations of the two probes in the waveguide are all arbitrary. As examples, expressions of mutual impedance for three specific cases are given and discussed.**

I. Introduction

The MUTUAL impedance between two probes vertically located on the broad wall of a rectangular waveguide was studied by Ittipiboon and Shafai [1] using the vector potential $\boldsymbol{A}$, and that between probes arbitrarily located in a rectangular waveguide was analyzed recently by the author [2] using the dyadic Green's function (DGF) $\boldsymbol{G}$. The above investigations are extremely useful in designing microwave circuits, various filters, and antennas with specific uses.

However, to the author's knowledge, the mutual impedance problem for a circular waveguide has not been considered. In this paper, the probe field distribution and mutual coupling in circular waveguide are studied in detail. The general formulas of mutual impedance between probes are given. In the derivation, the DGF and reaction theorem are used. The waveguide is semi-infinite. The reflection coefficient at the terminal plane ($z=0$) is Γ. The lengths, feeding points, and orientations of the two probes in the waveguide are all arbitrary.

II. The Dyadic Green's Function

The problem to be considered is shown in Fig. 1. Two probe antennas, arbitrarily oriented, are located in a circular waveguide. Suppose the radius of the waveguide is a and is filled with air (μ_0, ϵ_0). The DGF $\boldsymbol{G}$ of the first kind pertaining to the waveguide under study satisfies

$$\nabla \times \nabla \times \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') - k^2 \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = \boldsymbol{I}\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{1}$$

where $k=\omega\sqrt{\mu_0\epsilon_0}$ is the free-space wavenumber. The ex-

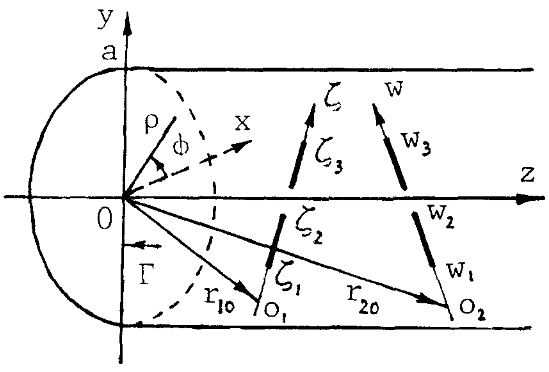

Fig 1. Two probes in a circular waveguide.

pression for $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ is given by [3], [4]

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = &-\frac{1}{k^2}\delta(\boldsymbol{r}-\boldsymbol{r}')\hat{z}\hat{z} \\ &+\frac{j}{4\pi}\sum_n\sum_m \frac{2-\delta_0}{\mu^2 k_\mu I_\mu}\left[\boldsymbol{M}_{n\mu}(\pm k_\mu)\boldsymbol{M}'_{n\mu}(\mp k_\mu)\right. \\ &\left.+\Gamma \boldsymbol{M}_{n\mu}(+k_\mu)\boldsymbol{M}'_{n\mu}(+k_\mu)\right] \\ &+\frac{j}{4\pi}\sum_n\sum_m \frac{2-\delta_0}{\lambda^2 k_\lambda I_\lambda}\left[\boldsymbol{N}_{n\lambda}(\pm k_\lambda)\boldsymbol{N}'_{n\lambda}(\mp k_\lambda)\right. \\ &\left.-\Gamma \boldsymbol{N}_{n\lambda}(+k_\lambda)\boldsymbol{N}'_{n\lambda}(+k_\lambda)\right], \qquad z \gtrless z'\end{aligned} \tag{2}$$

where $k_\mu=(k^2-\mu^2)^{1/2}$, $k_\lambda=(k^2-\lambda^2)^{1/2}$, $I_\mu=\frac{1}{2}a^2[1-(n/q_{nm})^2]J_n^2(q_{nm})$, $I_\lambda=\frac{1}{2}a^2 J_{n+1}^2(p_{nm})$, $\mu=q_{nm}/a$, and $\lambda=p_{nm}/a$. The quantities p_{nm} and q_{nm} are the mth roots of the nth-order Bessel function J_n and its derivative, J_n', respectively. The Kronecker delta $\delta_0=1$ for $n=0$ and $\delta_0=0$ for $n\neq 0$. In summing, n is from 0 to ∞ and m from 1 to ∞. For simplicity, the parallel subscripts e (even) and o (odd) on $\boldsymbol{M}$ and $\boldsymbol{N}$ in (2) are omitted, that is, $\boldsymbol{M}_{\substack{e\\o}n\mu}$ was simplified to $\boldsymbol{M}_{n\mu}$, and so on. $\boldsymbol{M}$ and $\boldsymbol{N}$ are written as follows:

$$\boldsymbol{M}_{n\mu}(\pm k_\mu) = \nabla \times \left[\hat{z}J_n(\mu\rho)\cos(n\phi-\phi_p)e^{\pm jk_\mu z}\right] \tag{3}$$

$$\boldsymbol{N}_{n\lambda}(\pm k_\lambda) = \frac{1}{k}\nabla\times\nabla\times\left[\hat{z}J_n(\lambda\rho)\cos(n\phi-\phi_p)e^{\pm jk_\lambda z}\right] \tag{4}$$

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where the angle $\phi_p = 0$ for the e component and $\phi_p = \pi/2$ for the o component. The general $\boldsymbol{G}$ is a sum of even $\boldsymbol{G}_e$ and odd $\boldsymbol{G}_o$ components.

The explicit expression for $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ is given by

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') = &-\frac{1}{k^2}\delta(\boldsymbol{r}-\boldsymbol{r}')\hat{z}\hat{z} + \frac{j}{4\pi k^2}\sum_n\sum_m(2-\delta_0)\\ &\cdot\{\hat{\rho}\hat{\rho}'[n^2k_1M_\rho M_\rho'(e_{1\mu}+\Gamma e_{2\mu})\\ &+\lambda^2k_2N_\rho N_\rho'(e_{1\lambda}+\Gamma e_{2\lambda})]\\ &+\hat{\rho}\hat{\phi}'[n\mu k_1M_\rho M_\phi'(e_{1\mu}+\Gamma e_{2\mu})\\ &-n\lambda k_2N_\rho N_\phi'(e_{1\lambda}+\Gamma e_{2\lambda})]\\ &+\hat{\rho}\hat{z}[\lambda k_3N_\rho N_z'(\pm e_{1\lambda}-\Gamma e_{2\lambda})]\\ &+\hat{\phi}\hat{\rho}'[n\mu k_1M_\phi M_\rho'(e_{1\mu}+\Gamma e_{2\mu})\\ &-n\lambda k_2N_\phi N_\rho'(e_{1\lambda}+\Gamma e_{2\lambda})]\\ &+\hat{\phi}\hat{\phi}'[\mu^2k_1M_\phi M_\phi'(e_{1\mu}+\Gamma e_{2\mu})\\ &+n^2k_2N_\phi N_\phi'(e_{1\lambda}+\Gamma e_{2\lambda})]\\ &+\hat{\phi}\hat{z}[nk_3N_\phi N_z'(\mp e_{1\lambda}+\Gamma e_{2\lambda})]\\ &+\hat{z}\hat{\rho}'[\lambda k_3N_zN_\rho'(\mp e_{1\lambda}-\Gamma e_{2\lambda})]\\ &+\hat{z}\hat{\phi}'[nk_3N_zN_\phi'(\pm e_{1\lambda}+\Gamma e_{2\lambda})]\\ &+\hat{z}\hat{z}[k_4N_zN_z'(e_{1\lambda}-\Gamma e_{2\lambda})]\}, \qquad z \gtrless z' \qquad (5)\end{aligned}$$

where

$$M_\rho = \frac{1}{\rho}J_n(\mu\rho)\sin(n\phi-\phi_p) \quad M_\phi = J_n'(\mu\rho)\cos(n\phi-\phi_p)$$

$$N_\rho = J_n'(\lambda\rho)\cos(n\phi-\phi_p) \quad N_\phi = \frac{1}{\rho}J_n(\lambda\rho)\sin(n\phi-\phi_p)$$

$$N_z = J_n(\lambda\rho)\cos(n\phi-\phi_p)$$

$$k_1 = \frac{k^2}{\mu^2 I_\mu k_\mu} \quad k_2 = \frac{k_\lambda}{\lambda^2 I_\lambda} \quad k_3 = \frac{j}{I_\lambda} \quad k_4 = \frac{\lambda^2}{I_\lambda k_\lambda}$$

$$e_{1i} = e^{\pm jk_i(z-z')} \quad e_{2i} = e^{jk_i(z+z')}, \quad i = \mu, \lambda.$$

The independent variables of M_ρ', M_ϕ', N_ρ', N_ϕ', and N_z' are ρ' and ϕ'. Note that all the second unit vectors (except $\hat{z}$) of the dyadic in (5) are primed ($\hat{\rho}', \hat{\phi}'$). The variable ϕ' should be used in coordinate transformation.

III. Field E_1 Radiated by Probe 1

The coordinate system of probe 1 is $O_1(\xi, \eta, \zeta)$, as shown in Fig. 1. ξ_1 and ζ_3 are the endpoints and ζ_2 is the feeding point. In the O system, the coordinates of point O_1 are $r_{10}(x_{10}, y_{10}, z_{10})$. Assume that the current distribution of probe 1 is given by

$$\boldsymbol{J}_1(\zeta) = \hat{\zeta}I_1\delta(\xi)\delta(\eta) \qquad (6a)$$

$$I_1 = \begin{cases} I_{10}\dfrac{\sin k(\zeta-\zeta_1)}{\sin k(\zeta_2-\zeta_1)}, & \zeta_1 \leq \zeta \leq \zeta_2 \\ I_{10}\dfrac{\sin k(\zeta_3-\zeta)}{\sin k(\zeta_3-\zeta_2)}, & \zeta_2 \leq \zeta \leq \zeta_3 \end{cases} \qquad (6b)$$

where I_{10} is the value of the current at the feeding point. The electric field radiated from probe 1 is

$$\boldsymbol{E}_1 = j\omega\mu_0\int_{v'}\boldsymbol{G}\cdot\boldsymbol{J}_1\,dv' = j\omega\mu_0\int_{\zeta_1}^{\zeta_3}\boldsymbol{G}\cdot I_1\,d\boldsymbol{\zeta} \qquad (7)$$

where $d\boldsymbol{\zeta} = \hat{x}\,dx' + \hat{y}\,dy' + \hat{z}\,dz'$. Assume that the direction cosines of $\hat{\zeta}$ are $s_1 = \cos\alpha_1$, $s_2 = \cos\beta_1$, $s_3 = \cos\gamma_1$. Hence, the parametric equations of the ζ axis are

$$x' = x_\zeta = s_1\zeta + x_{10} \qquad y' = y_\zeta = s_2\zeta + y_{10}$$
$$z' = z_\zeta = s_3\zeta + z_{10}. \qquad (8)$$

Through a tedious treatment (7), the principal value of the integral for $z > z'$ is found to be

$$\begin{pmatrix} E_\rho \\ E_\phi \\ E_z' \end{pmatrix} = \sum_n\sum_m(\phi)\begin{pmatrix} A_1M_xF_\mu(z) + A_2N_xF_\lambda(z) \\ A_1M_yF_\mu(z) + A_2N_yF_\lambda(z) \\ 0 + A_3N_zF_\lambda(z)\end{pmatrix} \qquad (9a)$$

whereas for $z < z'$ the principal value is

$$\begin{pmatrix} E_\rho \\ E_\phi \\ E_z' \end{pmatrix} = \sum_n\sum_m(\phi)\begin{pmatrix} B_1M_xF_\mu(+\Gamma z) + B_2N_xF_\lambda(+\Gamma z) \\ B_1M_yF_\mu(+\Gamma z) + B_2N_yF_\lambda(+\Gamma z) \\ 0 + B_3N_zF_\lambda(-\Gamma z)\end{pmatrix}. \qquad (9b)$$

In (9), a correction term E_z'' should be added to the term E_z' to yield a correct longitudinal component E_z; that is, $E_z = E_z' + E_z''$. The correction term is

$$\begin{aligned}\hat{z}E_z''(\rho,\phi,z) &= j\omega\mu_0\int_{\zeta_1}^{\zeta_3} -\frac{1}{k^2}\delta(\boldsymbol{r}-\boldsymbol{r}')\hat{z}\hat{z}\cdot I_1\,d\boldsymbol{\zeta}\\ &= \hat{z}j\omega\mu_0\int_{\zeta_1}^{\zeta_3} -\frac{1}{k^2}\delta(\boldsymbol{r}-\boldsymbol{r}')I_1\,dz'.\end{aligned}$$

Based on the distribution theory [5], [6], we have

$$\frac{1}{\rho'}\delta(\rho-\rho')\delta(\phi-\phi') = \sum_n\sum_m\frac{2-\delta_0}{2\pi I_\lambda}J_n(\lambda\rho)\cdot\cos(n\phi-\phi_p)J_n(\lambda\rho')\cos(n\phi'-\phi_p).$$

Hence the correction term is given by

$$E_z''(\rho,\phi,z) = -\frac{j\eta_0}{2k\pi}\sum_n\sum_m\frac{2-\delta_0}{I_\lambda}\frac{s_3}{|s_3|}I_1(\zeta)J_n(\lambda\rho_\zeta)\cdot J_n(\lambda\rho)\cos n(\phi-\phi_\zeta)\Big|_{\zeta=(z-z_{10})/s_3} \qquad (9c)$$

where $\rho_\zeta = (x_\zeta^2 + y_\zeta^2)^{1/2}$ and $\tan\phi_\zeta = y_\zeta / x_\zeta$. In (9),

$$M_x = nM_\rho \cos\phi - \mu M_\phi \sin\phi \qquad M_y = nM_\rho \sin\phi + \mu M_\phi \cos\phi$$

$$N_x = \lambda N_\rho \cos\phi + nN_\phi \sin\phi \qquad N_y = \lambda N_\rho \sin\phi - nN_\phi \cos\phi$$

$$F_i(\pm\Gamma z) = e^{-jk_i z} \pm \Gamma e^{jk_i z} \qquad F_i(z) = e^{jk_i z}, \qquad i = \mu, \lambda.$$

The elements of matrix (ϕ) are

$$\phi_{11} = \phi_{22} = \cos\phi \qquad \phi_{12} = -\phi_{21} = \sin\phi$$

$$\phi_{33} = 1 \qquad \text{other } \phi_{ij} = 0.$$

Other parameters of (9) are as follows:

$$\begin{cases} A_1 = -\dfrac{\eta_0(2-\delta_0)}{4\pi}(s_1 k_1 P_{Mx} + s_2 k_1 P_{My}) \\ A_2 = -\dfrac{\eta_0(2-\delta_0)}{4\pi}(s_1 k_2 P_{Nx} + s_2 k_2 P_{Ny} + s_3 k_3 P_{Nz}) \\ A_3 = -\dfrac{\eta_0(2-\delta_0)}{4\pi}(-s_1 k_3 P_{Nx} - s_2 k_3 P_{Ny} + s_3 k_4 P_{Nz}) \end{cases} \tag{10a}$$

$$\begin{cases} B_1 = -\dfrac{\eta_0(2-\delta_0)}{4\pi}(s_1 k_1 Q_{Mx} + s_2 k_1 Q_{My}) \\ B_2 = -\dfrac{\eta_0(2-\delta_0)}{4\pi}(s_1 k_2 Q_{Nx} + s_2 k_2 Q_{Ny} - s_3 k_3 Q_{Nz}) \\ B_3 = -\dfrac{\eta_0(2-\delta_0)}{4\pi}(s_1 k_3 Q_{Nx} + s_2 k_3 Q_{Ny} + s_3 k_4 Q_{Nz}) \end{cases} \tag{10b}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance of free space, and

$$P_{Mi} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 m_i(\zeta) f_\mu(+\Gamma\zeta)\, d\zeta, \qquad i = x, y \tag{11a}$$

$$Q_{Mi} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 m_i(\zeta) f_\mu(\zeta)\, d\zeta, \qquad i = x, y \tag{11b}$$

$$P_{Ni} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 n_i(\zeta) f_\lambda(+\Gamma\zeta)\, d\zeta, \qquad i = x, y \tag{11c}$$

$$P_{Nz} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 n_z(\zeta) f_\lambda(-\Gamma\zeta)\, d\zeta \tag{11d}$$

$$Q_{Ni} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 n_i(\zeta) f_\lambda(\zeta)\, d\zeta, \qquad i = x, y, z \tag{11e}$$

where

$$m_x(\zeta) = \frac{n x_\zeta}{\rho_\zeta^2} J_n(\mu\rho_\zeta)\sin(n\phi_\zeta - \phi_p) - \frac{\mu y_\zeta}{\rho_\zeta} J_n'(\mu\rho_\zeta)\cos(n\phi_\zeta - \phi_p) \tag{12a}$$

$$m_y(\zeta) = \frac{n y_\zeta}{\rho_\zeta^2} J_n(\mu\rho_\zeta)\sin(n\phi_\zeta - \phi_p) + \frac{\mu x_\zeta}{\rho_\zeta} J_n'(\mu\rho_\zeta)\cos(n\phi_\zeta - \phi_p) \tag{12b}$$

$$n_x(\zeta) = \frac{\lambda x_\zeta}{\rho_\zeta} J_n'(\lambda\rho_\zeta)\cos(n\phi_\zeta - \phi_p) + \frac{n y_\zeta}{\rho_\zeta^2} J_n(\lambda\rho_\zeta)\sin(n\phi_\zeta - \phi_p) \tag{12c}$$

$$n_y(\zeta) = \frac{\lambda y_\zeta}{\rho_\zeta} J_n'(\lambda\rho_\zeta)\cos(n\phi_\zeta - \phi_p) - \frac{n x_\zeta}{\rho_\zeta^2} J_n(\lambda\rho_\zeta)\sin(n\phi_\zeta - \phi_p) \tag{12d}$$

$$n_z(\zeta) = J_n(\lambda\rho_\zeta)\cos(n\phi_\zeta - \phi_p) \tag{12e}$$

$$f_i(\pm\Gamma\zeta) = e^{-jk_i z_\zeta} \pm \Gamma e^{jk_i z_\zeta}, \qquad i = \mu, \lambda \tag{13a}$$

$$f_i(\zeta) = e^{jk_i z_\zeta}, \qquad i = \mu, \lambda. \tag{13b}$$

IV. Mutual Impedance

In order to calculate mutual impedance, we must determine the tangent component of $\boldsymbol{E}_1$ along probe 2. With reference to Fig. 1, the coordinate system of probe 2 is $O_2(u, v, w)$; w_1 and w_3 are endpoints, and w_2 is the feeding point. The coordinates of point O_2 in the O system are (x_{20}, y_{20}, z_{20}). The current distribution of probe 2 is similar to that of probe 1 (eq. (6)), that is,

$$\boldsymbol{J}_2(w) = \hat{w} I_2 \delta(u)\delta(v) \tag{14a}$$

$$I_2 = \begin{cases} I_{20}\dfrac{\sin k(w - w_1)}{\sin k(w_2 - w_1)}, & w_1 \leqslant w \leqslant w_2 \\ I_{20}\dfrac{\sin k(w_3 - w)}{\sin k(w_3 - w_2)}, & w_2 \leqslant w \leqslant w_3. \end{cases} \tag{14b}$$

Assume that the direction cosines of $\hat{w}$ are $t_1 = \cos\alpha_2$, $t_2 = \cos\beta_2$, $t_3 = \cos\gamma_2$. The parametric equations of the w axis are given by

$$x_w = t_1 w + x_{20} \qquad y_w = t_2 w + y_{20} \qquad z_w = t_3 w + z_{20}. \tag{15}$$

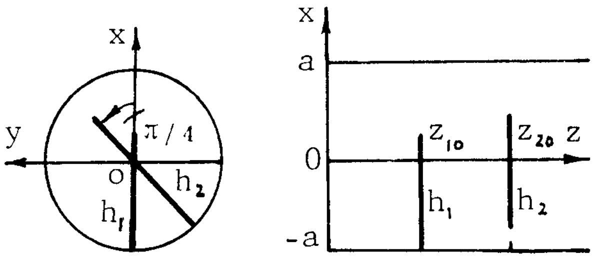

Fig. 2. Two probes meet at 45°.

The tangential component of $\boldsymbol{E}_1$ along probe 2 is given by

$$E_{1w} = \boldsymbol{E}_1 \cdot \hat{w} = (t_1 \cos\phi + t_2 \sin\phi) E_\rho + (-t_1 \sin\phi + t_2 \cos\phi) E_\phi + t_3 E_z. \quad (16)$$

Substituting for $\boldsymbol{E}_1$ from (9), we get, for $z > z'$,

$$E_{1w} = \sum_n \sum_m \{ t_1 [A_1 m_x(w) f_\mu(w) + A_2 n_x(w) f_\lambda(w)] + t_2 [A_1(w) m_y(w) f_\mu(w) + A_2 n_y(w) f_\lambda(w)] + t_3 [A_3 n_z(w) f_\lambda(w) + E_z''(\rho_w, \phi_w, z_w)] \} \quad (17a)$$

whereas for $z < z'$,

$$E_{1w} = \sum_n \sum_m \{ t_1 [B_1 m_x(w) f_\mu(+\Gamma w) + B_2 n_x(w) f_\lambda(+\Gamma w)] + t_2 [B_1 m_y(w) f_\mu(+\Gamma w) + B_2 n_y(w) f_\lambda(+\Gamma w)] + t_3 [B_3 n_z(w) f_\lambda(-\Gamma w) + E_z''(\rho_w, \phi_w, z_w)] \} \quad (17b)$$

where $m_i(x)$ $(i = x, y)$, $n_i(w)$ $(i = x, y, z)$ and $f_i(\pm \Gamma w)$, $f_i(w)$ $(i = \mu, \lambda)$ are similar to (12) and (13), respectively, except that $\zeta \to w$. In addition, $\rho_w = (x_w^2 + y_w^2)^{1/2}$ and $\tan\phi_w = y_w / x_w$. By the reaction concept, the mutual impedance between two probes is given by

$$M = -\frac{1}{I_{10} I_{20}} \int_{w_1}^{w_3} E_{1w} I_2 \, dw \quad (18)$$

where E_{1w} is given by (17), and I_2 is given by (14). The time factor used is $e^{-j\omega t}$. If we desire to adopt $e^{j\omega t}$, we need only replace j by $-j$ in all formulas.

V. Specific Cases

As examples, we discuss some useful cases. In the discussion below, suppose that the feeding points of two probes are coincident with O_1 and O_2, respectively. The heights are $h_1 = \zeta_3 - \zeta_2$ $(\zeta_1 = \zeta_2 = 0)$ and $h_2 = w_3 - w_2$ $(w_1 = w_2 = 0)$, respectively. What is more, assume that the probes in all the examples (except probes vertical to terminal plane) are parallel to the xy plane and through the center of the guide $(\rho = 0)$.

Example 1. Two Probes Meet at 45°

Suppose that the probe 1 is parallel to the x axis and probe 2 makes an angle of $\pi/4$ with the x axis, as shown in Fig. 2. The two are nonplanar skew probes. For probe 1, the coordinates of the feeding point are $(x_{10} = -a,\ y_{10} = 0,\ z_{10})$, the orientational parameters are $\alpha_1 = 0$, $\beta_1 = \gamma_1 = \pi/2$, $s_1 = 1$, $s_2 = s_3 = 0$. For probe 2, the parameters are $(x_{20} = y_{20} = -a/\sqrt{2},\ z_{20})$; $\alpha_2 = \beta_2 = \pi/4$, $\gamma_2 = \pi/2$, $t_1 = t_2 = 1/\sqrt{2}$, $t_3 = 0$. From (9), the field $\boldsymbol{E}_1$ radiated from probe 1 for $z > z_{10}$ is found to be

$$E_\rho = \sum_n \sum_m C_0 \left[-k_1 n^2 A_H \frac{1}{\rho} J_n(\mu\rho) \cos n\phi e^{jk_\mu z} - k_2 \lambda^2 A_E J_n'(\lambda\rho) \cos n\phi e^{jk_\lambda z} \right] \quad (19a)$$

$$E_\phi = \sum_n \sum_m C_0 \left[k_1 n \mu A_H J_n'(\mu\rho) \sin n\phi e^{jk_\mu z} + k_2 n \lambda A_E \frac{1}{\rho} J_n(\lambda\rho) \sin n\phi e^{jk_\lambda z} \right] \quad (19b)$$

$$E_z = \sum_n \sum_m C_0 k_3 \lambda A_E J_n(\lambda\rho) \cos n\phi e^{jk_\lambda z} \quad (19c)$$

where

$$C_0 = I_{10} \eta_0 (2 - \delta_0) / (4\pi \sin k h_1)$$

$$A_H = F_\mu(+\Gamma z_{10}) R_\mu(h_1) \qquad A_E = F_\lambda(+\Gamma z_{10}) R_\lambda(h_1)$$

and

$$R_\mu(h_1) = \frac{1}{k} \int_0^{h_1} \frac{1}{(\zeta - a|_o} J_n(\mu|\zeta - a|) \sin k(h_1 - \zeta) \, d\zeta \quad (20a)$$

$$R_\lambda(h_1) = \frac{1}{k} \int_0^{h_1} \frac{\zeta - a}{(\zeta - a|_e} J_n'(\lambda|\zeta - a|) \sin k(h_1 - \zeta) \, d\zeta \quad (20b)$$

where $(\zeta - a|_o = \zeta - a$ when n is zero or even and $(\zeta - a|_o = |\zeta - a|$ when n is odd. The field distribution for $z < z_{10}$ is

$$E_\rho = \sum_n \sum_m C_0 \left[-k_1 n^2 B_H \frac{1}{\rho} J_n(\mu\rho) \cos n\phi (e^{-jk_\mu z} + \Gamma e^{jk_\mu z}) - k_2 \lambda^2 B_E J_n'(\lambda\rho) \cos n\phi (e^{-jk_\lambda z} + \Gamma e^{jk_\lambda z}) \right] \quad (21a)$$

$$E_\phi = \sum_n \sum_m C_0 \left[k_1 n \mu B_H J_n'(\mu\rho) \sin n\phi (e^{-jk_\mu z} + \Gamma e^{jk_\mu z}) + k_2 n \lambda B_E \frac{1}{\rho} J_n(\lambda\rho) \sin n\phi (e^{-jk_\lambda z} + \Gamma e^{jk_\lambda z}) \right] \quad (21b)$$

$$E_z = \sum_n \sum_m - C_0 k_3 \lambda B_E J_n(\lambda\rho) \cos n\phi (e^{-jk_\lambda z} - \Gamma e^{jk_\lambda z}) \quad (21c)$$

where $B_H = F_\mu(z_{10}) R_\mu(h_1)$ and $B_E = F_\lambda(z_{10}) R_\lambda(h_1)$. In (19) and (21), the terms of A_H and B_H are H mode, and the terms of A_E and B_E are E mode. If probe 1 makes an angle of $\pi/4$ with the x axis, like probe 2 in Fig. 2, the electric fields are similar to (19) and (21), except that

$$\cos n\phi \to \cos n(\phi - \pi/4) \quad \text{and} \quad \sin n\phi \to \sin n(\phi - \pi/4).$$

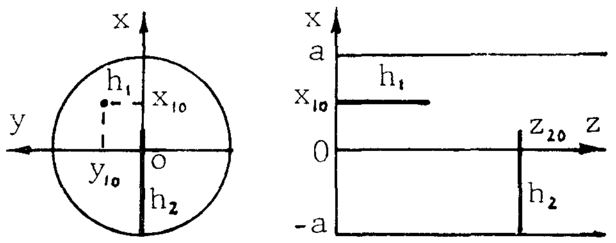

Fig. 3. One probe on the terminal wall and one probe on the cylindrical face.

The mutual impedance between probes in Fig. 2 is

$$M=\sum_n\sum_m D_0\cos\frac{n\pi}{4}\left\{k_1n^2R_\mu(h_1)R_\mu(h_2)\right.\cdot\left[e^{jk_\mu|z_{20}-z_{10}|}+\Gamma e^{jk_\mu(z_{20}+z_{10})}\right]+k_2\lambda^2R_\lambda(h_1)R_\lambda(h_2)\left.\left[e^{jk_\lambda|z_{20}-z_{10}|}+\Gamma e^{jk_\lambda(z_{20}+z_{10})}\right]\right\} \tag{22}$$

where $D_0=k\eta_0(2-\delta_0)/(4\pi\sin kh_1\sin kh_2)$, and $R_\mu(h_2)$ and $R_\lambda(h_2)$ are similar to (20), except that $h_1\to h_2$. The front terms (with subscript μ) and the back terms (with subscript λ) in (22) represent the contributions to the mutual impedance M from the H mode and the E mode, respectively. Setting $n=m=1$ in (22), we get the contributions to M from the H_{11} and E_{11}. When two probes meet at an angle ϕ_0, the mutual impedance between them is

$$M=\sum_n\sum_m D_0\cos n\phi_0\sim \tag{23}$$

where the $\sim$ represents all terms in the braces in (22).

Example 2. Two Probes Vertical and Parallel to the Terminal Wall, Respectively

Assume that the coordinates of the feeding point for probe 1 are $(x_{10}, y_{10}, 0)$. The orientational parameters are $\alpha_1=\beta_1=\pi/2$, $\gamma_1=0$; $s_1=s_2=0$, $s_3=1$. The parameters for probe 2 are $(x_{20}=-a,\ y_{20}=0,\ z_{20})$; $\alpha_2=0$, $\beta_2=\gamma_2=\pi/2$, $t_1=1$, $t_2=t_3=0$, as shown in Fig. 3. The field radiated from probe 1, for $z>h_1$ is

$$E_\rho=\sum_n\sum_m -C_0k_3\lambda C_EJ_n'(\lambda\rho)\cos n(\phi-\phi_{10})e^{jk_\lambda z} \tag{24a}$$

$$E_\phi=\sum_n\sum_m C_0k_3nC_E\frac{1}{\rho}J_n(\lambda\rho)\sin n(\phi-\phi_{10})e^{jk_\lambda z} \tag{24b}$$

$$E_z=\sum_n\sum_m -C_0k_4C_EJ_n(\lambda\rho)\cos n(\phi-\phi_{10})e^{jk_\lambda z} \tag{24c}$$

where $C_E=J_n(\lambda\rho_{10})T_\lambda(h_1)$, $\rho_{10}=(x_{10}^2+y_{10}^2)^{1/2}$, $\tan\phi_{10}=y_{10}/x_{10}$, and

$$T_\lambda(h_1)=\frac{1}{k}\int_0^{h_1}(e^{-jk_\lambda\zeta}-\Gamma e^{jk_\lambda\zeta})\sin k(h_1-\zeta)\,d\zeta. \tag{25}$$

In region $0<z<h_1$, the field in (x,y,z) is due to currents of sections $0z$ $(z>z')$ and zh_1 $(z<z')$; then (9a), (9b), and (9c) must be used simultaneously. The integral intervals are $0\to z$ for P_{Nz} and $z\to h_1$ for Q_{Nz}. Thus

$$E_\rho=\sum_n\sum_m C_0\left[-k_3\lambda C_E(z)J_n'(\lambda\rho)\cos n(\phi-\phi_{10})e^{jk_\lambda z}+k_3\lambda D_E(z)J_n'(\lambda\rho)\cos n(\phi-\phi_{10})\cdot(e^{-jk_\lambda z}+\Gamma e^{jk_\lambda z})\right] \tag{26a}$$

$$E_\phi=\sum_n\sum_m C_0\left[k_3nC_E(z)\frac{1}{\rho}J_n(\lambda\rho)\sin n(\phi-\phi_{10})e^{jk_\lambda z}-k_3nD_E(z)\frac{1}{\rho}J_n(\lambda\rho)\sin n(\phi-\phi_{10})\cdot(e^{-jk_\lambda z}+\Gamma e^{jk_\lambda z})\right] \tag{26b}$$

$$E_z'=\sum_n\sum_m C_0\left[-k_4C_E(z)J_n(\lambda\rho)\cos n(\phi-\phi_{10})e^{jk_\lambda z}-k_4D_E(z)J_n(\lambda\rho)\cos n(\phi-\phi_{10})\cdot(e^{-jk_\lambda z}-\Gamma e^{jk_\lambda z})\right] \tag{26c}$$

$$E_z''=\sum_n\sum_m -\frac{2}{k}C_0k_3J_n(\lambda\rho_{10})J_n(\lambda\rho)\cdot\cos n(\phi-\phi_{10})\sin k(h_1-z) \tag{26d}$$

where

$$C_E(z)=J_n(\lambda\rho_{10})\frac{1}{k}\int_0^z(e^{-jk_\lambda\zeta}-\Gamma e^{jk_\lambda\zeta})\sin k(h_1-\zeta)\,d\zeta \tag{27}$$

$$D_E(z)=J_n(\lambda\rho_{10})\frac{1}{k}\int_z^{h_1}e^{jk_\lambda\zeta}\sin k(h_1-\zeta)\,d\zeta. \tag{28}$$

Because probe 1 is vertical in relation to the terminal wall, the H_z component is not excited. Therefore there is only E mode in (24) and (26).

The mutual impedance between the probes in Fig. 3 is

$$M=\sum_n\sum_m D_0k_3\lambda T_\lambda(h_1)R_\lambda(h_2)J_n(\lambda\rho_{10})\cos n\phi_{10}e^{jk_\lambda z_{20}},\quad z_{20}>h_1. \tag{29}$$

When the feed point of probe 1 is situated at the center of the terminal plane (origin), $\rho_{10}=0$. The Bessel function $J_n(0)=0$ for $n\neq 0$ and $J_0(0)=1$. In this case, only the modes in which $n=0$ are excited; that is, the fields of the modes are independent of the azimuthal angle ϕ. Thus (24) reduces to

$$E_\rho=\sum_m C_0k_3\lambda C_EJ_1(\lambda\rho)e^{jk_\lambda z} \tag{30a}$$

$$E_z=\sum_m -C_0k_4C_EJ_0(\lambda\rho)e^{jk_\lambda z} \tag{30b}$$

whereas (26) reduces to

$$E_\rho=\sum_m C_0\left[k_3\lambda C_E(z)J_1(\lambda\rho)e^{jk_\lambda z}-k_3\lambda D_E(z)J_1(\lambda\rho)(e^{-jk_\lambda z}+\Gamma e^{jk_\lambda z})\right] \tag{31a}$$

$$E_z'=\sum_m C_0\left[-k_4C_E(z)J_0(\lambda\rho)e^{jk_\lambda z}-k_4D_E(z)J_0(\lambda\rho)(e^{-jk_\lambda z}-\Gamma e^{jk_\lambda z})\right] \tag{31b}$$

$$E_z''=\sum_m -\frac{2}{k}C_0k_3J_0(\lambda\rho)\sin k(h_1-z). \tag{31c}$$

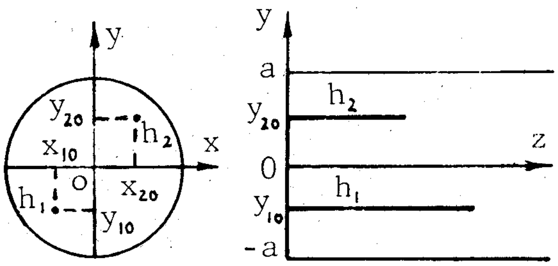

Fig. 4. Two probes on the terminal wall.

Hence (29) can be simplified as follows:

$$M=\sum_{m} D_0 k_3 \lambda T_\lambda(h_1) R_\lambda(h_2) e^{jk_\lambda z_{20}}. \qquad (32)$$

In C_0, C_E, $C_E(z)$, $D_E(z)$, and D_0 in (30)–(32), $2-\delta_0=1$ and $J_n(\lambda\rho_{10})=1$. $T_\lambda(h_1)$ is again given by (25) and $R_\lambda(h_2)$ reduces to

$$R_\lambda(h_2)=-\frac{1}{k}\int_0^{h_2}\frac{w-a}{|w-a|}J_1(\lambda|w-a|)\sin k(h_2-w)\,dw \qquad (33)$$

where $J_0'(\lambda|w-a|)=-J_1(\lambda|w-a|)$ is used.

Example 3. Two Probes Perpendicular to Terminal Wall

Assume that the coordinates of feeding points for the two probes are $(x_{10}, y_{10}, 0)$ and $(x_{20}, y_{20}, 0)$, respectively, as shown in Fig. 4. The orientational parameters of the two probes are $\alpha_1=\alpha_2=\beta_1=\beta_2=\pi/2$, $\gamma_1=\gamma_2=0$, $s_1=t_1=s_2=t_2=0$, $s_3=t_3=1$. When $h_1=h_2=h$, the mutual impedance is

$$M=\sum_n\sum_m \frac{1}{k^2}D_0\left[k_3\left(h-\frac{1}{2k}\sin 2kh\right)+k_4 H(h)\right] \cdot J_n(\lambda\rho_{10})J_n(\lambda\rho_{20})\cos n(\phi_{20}-\phi_{10}) \qquad (34)$$

where

$$H(h)=\int_{\zeta=0}^{h}\int_{w=0}^{h}\left[e^{jk_\lambda|w-\zeta|}-\Gamma e^{jk_\lambda(w+\zeta)}\right] \cdot \sin k(h-\zeta)\sin k(h-w)\,d\zeta\,dw. \qquad (35)$$

The integrating procedures for $H(h)$ are as follows:

$$\int_{\zeta=0}^{h}\int_{w=0}^{h}\sim d\zeta\,dw=\int_{\zeta=0}^{h}\left(\int_{w=0}^{\zeta}+\int_{w=\zeta}^{h}\right)\sim d\zeta\,dw$$

$$=\int_{w=0}^{h}\left(\int_{\zeta=0}^{w}+\int_{\zeta=w}^{h}\right)\sim d\zeta\,dw$$

where the $\sim$ represents the integrand in $H(h)$. In (34), the terms involving k_3 come from E_z''.

VI. Conclusions

The mutual impedance between two probes in a semi-infinite circular waveguide is analyzed. It can be seen that the mutual impedance is dependent not only on the probe lengths, orientations, and separation distance, but also on the guide size, the dielectric material, and the terminal reflection coefficient.

The constants k_μ and k_λ are real for traveling modes and imaginary for evanescent modes. Hence, it can be seen from (18) that when the reflection coefficient Γ is real, the contribution to the mutual resistance comes from traveling modes, while the contribution to the reactance is from the traveling and evanescent modes. When Γ is complex, the mutual resistance and reactance will be dependent on all modes.

The lowest mode in circular waveguide is H_{11}. Suppose that the H_{11} is the only traveling mode and that the higher order modes are evanescent modes that decay exponentially with the distance. Consider $\Gamma=-1$. Equations (22) and (23) include the resistance and reactance because the contribution from the H_{11} mode exists. Equations (29), (32), and (34) include only the reactance because there is no contribution from the H_{11}. In overmoded guide, the mutual impedance from the higher order modes will be of interest.

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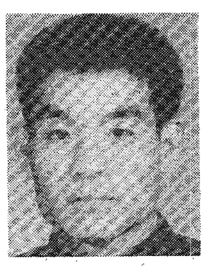

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